

## 4.4. Dimensional Analysis Method

### 4.4.1 Basic Concept

#### \* Dimensions

A measurable quantity has a "dimension" associated with it. For example:

length dimension	[L]
time dimension	[T]
mass dimension	[M]
temperature dimension	[K]

(independent of the unit to be used)

#### \* Primary Dimensions

The dimensions one chooses as the basis to measure other quantities are primary dimensions (which are associated with dimensional independent variables).

The dimensions of other quantities can be expressed as a product of powers of the primary dimensions.

e.g. For heat transfer problems one can choose:

[L], [T], [M], [K] as the basis (primary dimensions)

therefore:  $\left\{ \begin{array}{l} \text{heat flux } q \text{ has a dimension: } [MT^{-3}] \\ \text{heat diffusivity } \alpha \text{ has a dimension: } [L^2T^{-1}] \\ \text{thermal conductivity } k \text{ has a dimension: } [MLT^{-3}K^{-1}] \end{array} \right.$

Note:  $\left\{ \begin{array}{l} \text{the unit for } q: \frac{W}{m^2} \\ \text{the unit for } \alpha: \frac{m^2}{s} \\ \text{the unit for } k: \frac{W}{m \cdot K} \end{array} \right.$

( $1W = 1 \text{ kg} \cdot \frac{m^2}{s^3}$ )

### \* Dimensional analysis

The objective of dimensional analysis is to group several variables (including parameters) together to form a new variable that is nondimensional.

e.g.  $\alpha$  has a dimension  $[L^2 T^{-1}]$

We can group  $\alpha$  with  $x$  and  $t$ :

$$\Pi_\alpha \equiv \frac{\alpha}{x^2/t} \longrightarrow \Pi_\alpha \text{ is nondimensional}$$

### \* Dimensional Analysis of a function

A function relates one dependent variable (e.g.  $T$ ) to independent variables as well as parameters.

e.g.  $T = f(x, t, \alpha, k, f_0)$

$f_0$  may be given as the B.C.

By choosing appropriate dimensional independent variables (including parameters), the original function can be transformed to a new function that contains only nondimensional variables/parameters. The number of variables/parameters is also reduced by the number of dimensional independent variables/parameters.

Step 1: Find the dimension for each variable/parameter in terms of primary dimensions, and determine the number of primary dimensions (i.e., the number of dimensional independent variables/parameters).

- Step 2. Choose appropriate dimensional independent variables/parameters according to given function or problem.
- Step 3. Through dimensional analysis, the remaining variables/parameters can be expressed in terms of the selected dimensional independent variables/parameters to form nondimensional variables/parameters.
- Step 4. The original function is transformed to a new function that only contains nondimensional variables/parameters.

Example:  $T = f(x, t, \alpha, k, \rho_0)$

$$\begin{array}{cccccc} \uparrow & & \uparrow & \uparrow & \uparrow & \uparrow \\ [K] & & [L] & [T] & [L^3 T^{-1}] & [M T^{-3}] \end{array}$$

There are 4 dimensional independent variables/parameters.  
(4 primary dimensions)

Choose  $\{x, t, \rho_0, k\}$  as the 4 dimensional independent variables/parameters (selection is not unique).

The remaining variables/parameters are  $T$  and  $\alpha$ , which can be expressed in terms of  $\{x, t, \rho_0, k\}$  to form nondimensional variables/parameters.

$$\left\{ \begin{array}{l} \pi_T \equiv \frac{T}{\frac{\rho_0}{k} x} \\ \pi_\alpha \equiv \frac{\alpha}{x^2/t} \end{array} \right.$$

The original function  $T = f(x, t, \alpha, k, \rho_0)$  is reduced to:

$$\pi_T = F(\pi_x), \text{ i.e.: } \left( \frac{T}{\frac{\rho_0}{k} x} \right) = F\left( \frac{\alpha}{x^2/t} \right)$$

equivalently:  $T = \frac{\rho_0 x}{k} F\left( \frac{x}{\sqrt{\alpha t}} \right)$

\*  $\pi$  theorem: — general case

Assume we have a function of  $n$  variables.

$$Q = f(a_1, a_2, a_3, \dots, a_n)$$

There are  $m$  dimensional independent variables:

$$\{a_1, a_2, \dots, a_m\}$$

So the dimensions of other variables can be expressed as a product of powers of the dimensions of the  $m$  dimensional independent variables:

$$\begin{cases} [a_{m+1}] = [a_1]^{r_{11}} [a_2]^{r_{12}} \dots [a_m]^{r_{1m}} \\ [a_{m+2}] = [a_1]^{r_{21}} [a_2]^{r_{22}} \dots [a_m]^{r_{2m}} \\ \vdots \\ [a_n] = [a_1]^{r_{n1}} [a_2]^{r_{n2}} \dots [a_m]^{r_{n-m,m}} \end{cases}$$

and:  $[Q] = [a_1]^{r_1} [a_2]^{r_2} \dots [a_m]^{r_m}$

Organize original variables into nondimensional form:

$$\left\{ \begin{array}{l} \pi_1 = \frac{a_{m+1}}{a_1^{r_{11}} a_2^{r_{12}} \dots a_m^{r_{1m}}} \\ \pi_2 = \frac{a_{m+2}}{a_1^{r_{21}} a_2^{r_{22}} \dots a_m^{r_{2m}}} \\ \vdots \\ \pi_{n-m} = \frac{a_n}{a_1^{r_{n-m,1}} a_2^{r_{n-m,2}} \dots a_m^{r_{n-m,m}}} \end{array} \right.$$

and:  $\pi = \frac{a}{a_1^{r_1} a_2^{r_2} \dots a_m^{r_m}}$

The original function is transformed into:

$$\pi = \bar{F}(\pi_1, \pi_2, \dots, \pi_{n-m})$$

( $n-m$  variables/parameters)

#### 4.4.2. Applications to solve heat conduction problems.

\* Example. Consider a 1D conduction problem of a semi-infinite medium, with constant heat flow at one end:

$$\boxed{\begin{array}{l} \frac{\partial^2 T(x,t)}{\partial x^2} = \frac{1}{\alpha} \frac{\partial T(x,t)}{\partial t} \\ \text{B.C. } \left\{ \begin{array}{l} -k \frac{\partial T}{\partial x} \Big|_{x=0} = q_0 \\ T \Big|_{x \rightarrow \infty} = 0 \end{array} \right. \\ \text{I.C. } T \Big|_{t=0} = 0 \end{array}}$$

Dimensional analysis:

$$T = f(x, t, \alpha, k, \rho_0) \quad (\text{as discussed before})$$

choose 4 dimensional independent variables/parameters:  $\begin{Bmatrix} x \\ t \\ \rho_0 \\ k \end{Bmatrix}$

$$\text{Then: } \begin{cases} \Pi_T \equiv \frac{T}{\frac{\rho_0}{k} x} \\ \Pi_\alpha \equiv \frac{\alpha}{x^2/t} \end{cases}$$

$$\text{so: } \frac{T}{\left(\frac{\rho_0}{k} x\right)} = F\left(\frac{x}{\sqrt{\alpha t}}\right) \quad \text{Nondimensional form.}$$

$$\text{i.e. } T = \frac{\rho_0}{k} x F\left(\frac{x}{\sqrt{\alpha t}}\right)$$

$$\text{define } \frac{x}{\sqrt{\alpha t}} \equiv \xi$$

$$\text{so: } T = \frac{\rho_0}{k} x F(\xi)$$

Using the conduction equation:  $\Rightarrow$  PDE  $\rightarrow$  ODE

$$\frac{\partial}{\partial x} \left[ F(\xi) \frac{\rho_0}{k} + \frac{\rho_0 x}{k} F'(\xi) \frac{1}{\sqrt{\alpha t}} \right] = \frac{\partial T}{\partial (\alpha t)}$$

$$\frac{\rho_0}{k} F'(\xi) \frac{1}{\sqrt{\alpha t}} + \frac{\rho_0}{k} \frac{1}{\sqrt{\alpha t}} F'(\xi) + \frac{\rho_0 x}{k} \frac{1}{\alpha t} F''(\xi) = \frac{\rho_0 x}{k} F'(\xi) \frac{-\frac{x}{2}}{(\alpha t)^{3/2}}$$

$$F''(\xi) = - \left[ \frac{2}{\xi} + \frac{\xi}{2} \right] F'(\xi)$$

$$F(\xi) = -C \left( \frac{e^{-\xi^2/4}}{\xi} - \frac{1}{2} \int_{\xi}^{\infty} e^{-\xi^2/4} d\xi \right)$$

$$\text{From B.C. } \Rightarrow C = -\frac{2}{\sqrt{\pi}}$$

$$\text{Then: } F(\xi) = \frac{2}{\sqrt{\pi}} \left( \frac{e^{-\xi^2/4}}{\xi} - \frac{1}{2} \int_{\xi}^{\infty} e^{-\xi^2/4} d\xi \right)$$

$$\text{and } T(x,t) = \frac{\rho_0}{k} \sqrt{\frac{\alpha t}{\pi}} \left( e^{-\frac{x^2}{4\alpha t}} - \frac{x}{\sqrt{\alpha t}} \int_{\frac{x}{\sqrt{\alpha t}}}^{\infty} e^{-\xi^2/4} d\xi \right)$$